4-5 *E*-field Computation using Gauss's Law

Reading Assignment: *pp. 98-107*

Q:

A:

1. HO: Spherically Symmetric Charge Densities

   Example: Using Gauss's Law to Determine the Electric Field

2. HO: Cylindrically Symmetric Charge Densities

1. **A hollow, charged cylinder (4.29)**
2. **Balanced, coaxial cylinders (4.31)**
3. **A Uniformly charged sphere (4.34)**
4. **A hollow, charged sphere (4.37)**

$\rho_{v1}(\bar{r})$ creates $\mathbf{E}_1(\bar{r})$,

$\rho_{v2}(\bar{r})$ creates $\mathbf{E}_2(\bar{r})$,

$\rho_v(\bar{r}) = \rho_{v1}(\bar{r}) + \rho_{v2}(\bar{r})$ creates $\mathbf{E}(\bar{r}) = \mathbf{E}_1(\bar{r}) + \mathbf{E}_2(\bar{r})$.

Spherically Symmetric Charge Densities

Consider volume charge densities $\rho_v(\bar{\mathrm{r}})$ that are functions of spherical coordinate r **only**, e.g.:

$$\rho_v(\bar{\mathrm{r}}) = \frac{1}{r^2} \qquad \text{or} \qquad \rho_v(\bar{\mathrm{r}}) = e^{-r}$$

We call these types of charge densities **spherically symmetric**, as the charge density changes as a function of the distance from the origin only (i.e., is independent of coordinates θ or ϕ).

As a result, the charge distribution in this case looks sort of like a "**fuzzy ball**", centered at the origin!

Using the point form of Gauss's Law, we find the resulting static electric field **must** have the form:

$$\mathbf{E}(\bar{\mathrm{r}}) = E(r)\,\hat{a}_r \qquad (\text{ for spherically symmetric } \rho_v(\bar{\mathrm{r}})\ )$$

Think about what **this** says. It states that the resulting static electric field from a spherically symmetric charge density is:

* A function of spherical coordinate r **only**.

* Points in the direction $\hat{a}_r$ (i.e., away from the origin at every point).

As a result, we can use the **integral form** of Gauss's Law to determine the **specific scalar** function $E(r)$ resulting from some **specific**, spherically symmetric charge density $\rho_v(\bar{r})$.

Recall the integral form of **Gauss's Law**:

$$\oint\!\!\!\oint_S \mathbf{E}(\bar{r}) \cdot \overline{ds} = \frac{Q_{enc}}{\varepsilon_0}$$

$$= \frac{1}{\varepsilon_0} \iiint_V \rho_v(\bar{r})\, dv$$

Consider now a surface S that is a **sphere** with radius r, centered at the origin. We call this surface the **Gaussian Surface** for spherically symmetric charge densities.

To we why, we integrate over this Gaussian surface and find:

$$\oint\!\!\!\oint_S \mathbf{E}(\bar{r}) \cdot \overline{ds} = \int_0^{2\pi}\int_0^{\pi} \mathbf{E}(\bar{r}) \cdot \hat{a}_r \;\; r^2 \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} E(r)\, \hat{a}_r \cdot \hat{a}_r \;\; r^2 \sin\theta \, d\theta \, d\phi$$

$$= E(r)\, r^2 \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi$$

$$= 4\pi r^2 E(r)$$

Therefore, from Gauss's Law, we get:

$$4\pi\, r^2 E(r) = \frac{Q_{enc}}{\varepsilon_0}$$

Rearranging, we find that the function E(*r*) is:

$$E(r) = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}$$

The enclosed charge Q_{enc} can be determined for a **spherically symmetric** distribution (a function of *r* only!) as:

$$\begin{aligned} Q_{enc} &= \iiint_V \rho_v(\overline{\mathrm{r}})\, dv \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho_v(r')\, r'^2 \sin\theta\, dr' d\theta d\phi \\ &= 4\pi \int_0^{r} \rho_v(r')\, r'^2 dr' \end{aligned}$$

Therefore, we find that the static electric field produced by a **spherically symmetric** charge density is $\mathbf{E}(\overline{\mathrm{r}}) = E(\mathrm{r})\hat{a}_r$, where the scalar function E(*r*) is:

$$\begin{aligned} E(r) &= \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} \\ &= \frac{1}{\varepsilon_0\, r^2}\int_0^{r} \rho_v(r')\; r'^2 dr' \end{aligned}$$

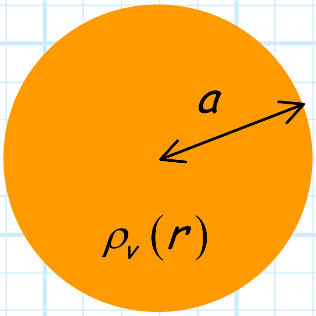
Or, more specifically, we find that the static electric field produced by some **spherically symmetric** charge density $\rho_v(\bar{\mathrm{r}})$ is:

$$\begin{aligned}\mathbf{E}(\bar{\mathrm{r}}) &= \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\hat{a}_r \\ &= \frac{\hat{a}_r}{\varepsilon_0 r^2}\int_0^r \rho_v(r')\, r'^2 dr'\end{aligned}$$

Thus, for a **spherically symmetric** charge density, we can find the resulting electric field **without** the difficult integration and evaluation required by **Coulomb's Law**!

Example: Using Gauss's Law to Determine the Electric Field

Consider a "cloud" of charge with **radius** *a* and centered at the origin, described by volume charge density:



$$\rho_v(\bar{r}) = \begin{cases} \dfrac{1}{r} & r < a \\ 0 & r > a \end{cases}$$

Q: *What electric field* $\mathbf{E}(\bar{r})$ *is produced by this charge ?*

A: We **could** use Coloumb's Law to solve this, but note that this is a **spherically symmetric** charge density! As a result, we can find the electric field much easier using **Gauss's Law**.

Recall that **spherically symmetric** charge densities produce an electric field:

$$\mathbf{E}(\bar{\mathbf{r}}) = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}\,\hat{a}_r$$

$$= \frac{\hat{a}_r}{\varepsilon_0 r^2}\int_0^r \rho_v(r')\,r'^2 dr'$$

Evaluating the **integral**, we need to consider **two** cases: one where r (i.e., the radius of the **Gaussian surface**) is **less than** cloud radius a (for evaluating the field **within** the charge cloud), and the second where r is **greater than** cloud radius a (for evaluating the field **outside** the charge cloud).

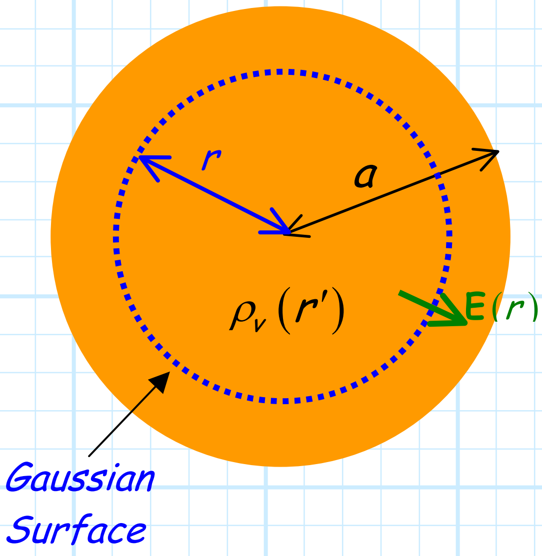
For $r < a$:

$$\int_0^r \rho_v(r')\,r'^2 dr' = \int_0^r \left(\frac{1}{r'}\right) r'^2 dr'$$

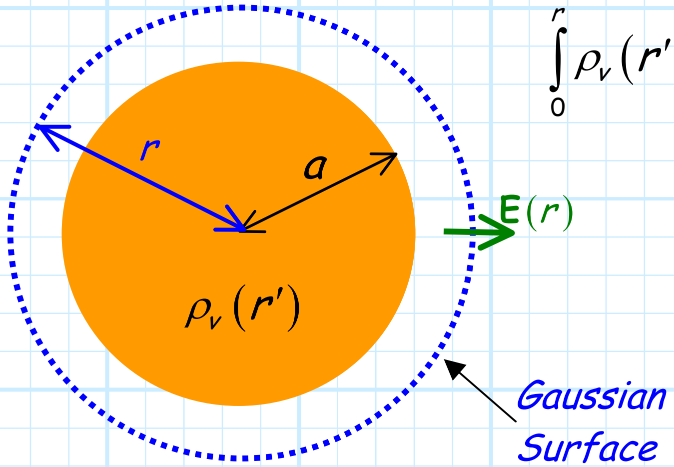
$$= \int_0^r r' dr'$$

$$= \left.\frac{r'^2}{2}\right|_0^r$$

$$= \frac{r^2}{2}$$



And if **r** > *a:*



$$\begin{aligned}\int_0^r \rho_v(r')\, r'^2 dr' &= \int_0^a \left(\frac{1}{r'}\right) r'^2 dr' + \int_a^r 0\, r'^2 dr' \\ &= \int_0^a r' dr' + 0 \\ &= \left.\frac{r'^2}{2}\right|_0^a \\ &= \frac{a^2}{2}\end{aligned}$$

Therefore, the **electric field** produced by this charge is:

$$\mathbf{E}(\bar{\mathrm{r}}) = \frac{\hat{a}_r}{\varepsilon_0 r^2}\int_0^r \rho_v(r')\, r'^2 dr'$$

$$= \begin{cases} \dfrac{\hat{a}_r}{\varepsilon_0 r^2}\left(\dfrac{r^2}{2}\right) & r < a \\[2ex] \dfrac{\hat{a}_r}{\varepsilon_0 r^2}\left(\dfrac{a^2}{2}\right) & r > a \end{cases}$$

$$= \begin{cases} \dfrac{\hat{a}_r}{2\varepsilon_0} & r < a \\[2ex] \dfrac{\hat{a}_r}{2\varepsilon_0}\left(\dfrac{a^2}{r^2}\right) & r > a \end{cases}$$

Note the resulting electric field behaves **as expected**. The field points in the direction $\hat{a}_r$ (i.e., points away from the origin). It is likewise independent of θ or ϕ (i.e., **spherically symmetric**).

Note also that the magnitude of the field outside of the cloud **diminishes** as $1/r^2$. This **makes sense**! Do **you** see why?

Cylindrically Symmetric Charge Densities

Consider the volume charge densities $\rho_v(\bar{\mathbf{r}})$ that are functions of cylindrical coordinate ρ **only**, e.g.:

$$\rho_v(\bar{\mathbf{r}}) = \frac{1}{\rho^2} \qquad \text{or} \qquad \rho_v(\bar{\mathbf{r}}) = e^{-\rho}$$

We call these types of charge densities **cylindrically symmetric**, as the charge density changes as a function of the distance from the *z*-axis only (i.e., is independent of coordinates ϕ or z).

As a result, the charge distribution in this case looks sort of like a "**fuzzy cylinder**", centered around the *z*-axis!

Using the point form of Gauss's Law, we find the resulting static electric field **must** have the form:

$$\mathbf{E}(\bar{\mathbf{r}}) = E(\rho)\ \hat{a}_\rho \qquad (\text{ for cylindrically symmetric } \rho_v(\bar{\mathbf{r}})\ )$$

Think about what **this** says. It states that the resulting static electric field from a cylindrically symmetric charge density is:

* A function of cylindrical coordinate ρ **only.**

* Points in the direction $\hat{a}_\rho$ (i.e., away from the *z*-axis) at every point.

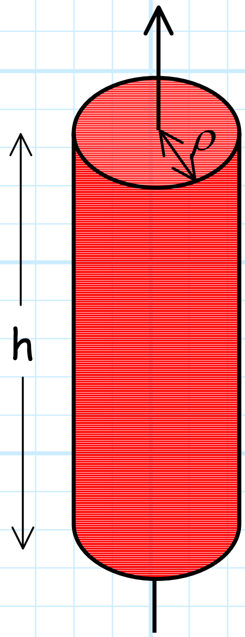
As a result, we can use the **integral form** of Gauss's Law to determine the specific **scalar** function E(ρ) resulting from some **specific**, cylindrically symmetric charge density $\rho_v(\bar{r})$.

Recall the integral form of **Gauss's Law**:

$$\oiint_S \mathbf{E}(\bar{r}) \cdot \overline{ds} = \frac{Q_{enc}}{\varepsilon_0}$$

$$= \frac{1}{\varepsilon_0} \iiint_V \rho_v(\bar{r})\, dv$$

Say surface S is a cylinder with radius ρ, centered along the *z*-axis. Additionally, this cylinder has a finite length *h*. We call this surface a **Gaussian Surface** for this problem.



We find that, if $\rho_v(\bar{\mathbf{r}})$ is **cylindrically symmetric**, then:

$$\oint\!\!\oint_S \mathbf{E}(\bar{\mathbf{r}})\cdot\overline{ds} = \int_{-h/2}^{h/2}\int_0^{2\pi} E(\rho)\,\hat{a}_\rho \cdot \hat{a}_\rho\ \rho\, d\phi\, dz \quad \text{side}$$

$$+\int_0^{2\pi}\int_0^{\rho} E(\rho')\,\hat{a}_\rho \cdot \hat{a}_z\ \rho'\, d\rho'\, d\phi \quad \text{top}$$

$$-\int_0^{2\pi}\int_0^{\rho} E(\rho')\,\hat{a}_\rho \cdot \hat{a}_z\ \rho'\, d\rho'\, d\phi \quad \text{bottom}$$

$$= E(\rho)\,\rho \int_{-h/2}^{h/2}\int_0^{2\pi} d\phi\, dz$$

$$= h\,2\pi\rho\ E(\rho)$$

Therefore, from **Gauss's Law**, we get:

$$h\,2\pi\,\rho\,E(\rho) = \frac{Q_{enc}}{\varepsilon_0}$$

Rearranging, we find that the **scalar** function $E(\rho)$ is:

$$E(\rho) = \frac{Q_{enc}}{2\pi\varepsilon_0\, h\rho}$$

The enclosed charge Q_{enc} can be determined for a **cylindrically symmetric** distribution as:

$$
\begin{aligned}
Q_{enc} &= \iiint_V \rho_v(\bar{\mathbf{r}})\, dv \\
&= \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^{\rho} \rho_v(\rho')\ \rho'\, d\rho'\, d\phi\, dz \\
&= 2\pi h \int_0^{\rho} \rho_v(\rho')\, \rho'\, d\rho'
\end{aligned}
$$

Therefore, we find that the static electric field produced by a **cylindrically symmetric** charge density is $\mathbf{E}(\bar{\mathbf{r}}) = E(\rho)\,\hat{a}_\rho$, where the **scalar** function $E(\rho)$ is:

$$
\begin{aligned}
E(\rho) &= \frac{Q_{enc}}{2\pi\varepsilon_0 h \rho} \\
&= \frac{1}{\varepsilon_0 \rho} \int_0^{\rho} \rho_v(\rho')\, \rho'\, d\rho'
\end{aligned}
$$

Or, more specifically, we find that the static electric field produced by some **cylindrically symmetric** charge density $\rho_v(\bar{\mathbf{r}})$ is:

$$
\begin{aligned}
\mathbf{E}(\bar{\mathbf{r}}) &= \frac{Q_{enc}}{2\pi\varepsilon_0 h \rho}\hat{a}_\rho \\
&= \frac{\hat{a}_\rho}{\varepsilon_0 \rho} \int_0^{\rho} \rho_v(\rho')\, \rho' d\rho'
\end{aligned}
$$

Thus, for a **cylindrically symmetric** charge density, we can find the resulting electric field **without** the difficult integration and evaluation required by **Coulomb's Law**!